Multilayer Analytic Element Modeling of Radial Collector Wells

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(1) Yep, same guy
Outline

• How Multilayer Systems Ought To Work
• How Radial Collector Wells In A Multilayer Aquifer Ought To Work
• Hooray, a Use For that 3D Model!
• Proof that Bigger Isn’t Always Better
What?

- Separate aquifer into individually homogeneous horizontal layers
- One layer contains RCW
- Arms of RCW modeled with multiaquifer line sinks
- RCW complexities incorporated
Why?

- Horizontal flow modeled analytically
- Vertical stratification / anisotropy
- BCs such as friction, head losses
- Can be combined with regional 2D model
Analytic Multilayer Approach

- Divide aquifer vertically into $N$ individually homogeneous layers (D-F approx in each)
- Transmission between layers governed by:
  - vertical discharge from center to center
  - vertical hydraulic conductivity
  - vertical resistance
- Net result: system of $N$ differential equations.....
Multilayer System Equations

\[ q_{z,i} = \frac{h_i - h_{i-1}}{c_i} \]

\[ c_i = \frac{H_{i-1}}{2k_{v,i-1}} + \frac{H_i}{2k_{v,i}} \]

\[ \nabla^2 h_i = \frac{q_{z,i} - q_{z,i+1}}{T_i} \]

\[ \nabla^2 h_i = \frac{h_i - h_{i-1}}{c_i T_i} - \frac{h_{i+1} - h_i}{c_{i+1} T_i} \]
The Radial Collector Well

- The RCW is in one of the N layers
- Thickness of this layer = diameter of RCW arms
- Each arm modeled by M multiaquifer line sinks (constant strength)
- Each sink of length L
- Discharge of arm j:

\[ Q_j = \sum_{m=1}^{M} \sigma_{m,j} L_{m,j} \]
A Call To Arms

\[ \sigma = q \frac{2 \pi R}{c} \]

\[ q = \frac{h - \phi}{c} \]

\[ \Rightarrow \sigma = \frac{h - \phi}{d} \]

\[ d = \frac{c}{2 \pi R} \]

\( \phi \): head inside arm [L]

\( h \): head outside arm [L]

\( c \): resistance to inflow [T]
Head Loss

• Quantified by comparison of conditions at the centers of adjacent line sinks

• \( \phi_m \): head at center of line sink \( m \)

\[
\Delta \phi_M = \phi_M - \phi_c
\]

\[
\Delta \phi_m = \phi_m - \phi_{m+1}
\]

Recall: \( \sigma_m = (h_m - \phi_m) / d \)
For adjacent line sinks \((m < M)\):

\[
\sigma_m = \frac{(h_m - \phi_m)}{d}
\]

\[
\sigma_{m+1} = \frac{(h_{m+1} - \phi_{m+1})}{d}
\]

Combine:

\[
d(\sigma_m - \sigma_{m+1}) = (h_m - h_{m+1}) - \Delta\phi_m
\]

Or

\[
(h_m - h_{m+1}) - d(\sigma_m - \sigma_{m+1}) = \Delta\phi_m
\]
For link sink $M$

$$\sigma_M = \frac{(h_M - \phi_M)}{d} \quad ; \quad \Delta \phi_M = \phi_M - \phi_c$$

$$\Rightarrow$$

$$d\sigma_M = h_M - (\Delta \phi_M + \phi_c)$$

$$h_M - d(\sigma_M) - \phi_c = \Delta \phi_M$$
Finding $\Delta \phi_m$ (for $m < M$)

- Darcy$^{(2)}$-Weisbach for $m < M$

$$\Delta \phi_m = f \frac{l_m V_m^2}{D \ 2g}$$

$$Q_m = \pi R^2 V_m$$

$$V_m^2 = Q_m^2 / \pi^2 R^4$$

$$\Delta \phi_m = f \frac{l_m Q_m^2}{2\pi^2 R^5 \ 2g}$$

(2) oh, nevermind
**Finding \( \Delta \phi_m \) (for \( m = M \))**

- For \( m = M \):
  
  (segment M)

  \[
  \Delta \phi_M = f \frac{l_M}{D} \frac{V_M^2}{2g}
  \]

  \( V_m^2 = Q_m^2 / \pi^2 R^4 \) ; \( K_L = 1.0 \)

  \[
  \Delta \phi_M = \left( f \frac{l_M}{2R} + 1 \right) \frac{Q_M^2}{2g \pi^2 R^4}
  \]

  (into caisson)

  \[
  \Delta \phi_C = K_L \frac{V_M^2}{2g}
  \]
Good Grief, Now We Need $Q_m$

- For $m < M$ ...

- $S_m$: total discharge at center of sink $m$
  \[ S_m = \sigma_1 L_1 + ... + \sigma_{m-1} L_{m-1} + (\sigma_m L_m)/2 \]
  \[ S_{m+1} = \sigma_1 L_1 + ... + \sigma_m L_m + (\sigma_{m+1} L_{m+1})/2 \]

- $Q_m$: average discharge, $\frac{(S_m + S_{m+1})}{2}$
  \[ Q_m = \sigma_1 L_1 + ... + \sigma_{m-1} L_{m-1} + \]
  \[ (3\sigma_m L_m + \sigma_{m+1} L_{m+1})/4 \]
$Q_m$ for $m = M$

- Compare center of sink $M$ to end of arm at caisson:

  \[ S_M = \sigma_1 L_1 + \ldots + \sigma_{M-1} L_{M-1} + \left( \frac{\sigma_M L_M}{2} \right) \]

  \[ S_c = \sigma_1 L_1 + \ldots + \sigma_M L_M \]

- $Q_M$: average discharge, \( \frac{(S_M + S_c)}{2} \)

  \[ Q_M = \sigma_1 L_1 + \ldots + \sigma_{M-1} L_{M-1} + \frac{3\sigma_M L_M}{4} \]
Recap: For $m < M$,

$$(h_m - h_{m+1}) - d(\sigma_m - \sigma_{m+1}) = \Delta \phi_m \quad (**)$$

where

$$\Delta \phi_m = f \frac{l_m}{2\pi^2R^5} \frac{Q_m^2}{2g}$$

and

$$Q_m = \sigma_1L_1 + \ldots + \sigma_{m-1}L_{m-1} + (3\sigma_mL_m + \sigma_{m+1}L_{m+1})/4$$
Recap: For \( m = M \),

\[
h_M - d(\sigma_M) - \phi_c = \Delta \phi_M \quad (**)
\]

where

\[
\Delta \phi_M = \left( f \frac{l_M}{2R} + 1 \right) \frac{Q_M^2}{2g\pi^2 R^4}
\]

and

\[
Q_M = \sigma_1 L_1 + \ldots + \sigma_{m-1} L_{m-1} + 3\sigma_M L_M / 4
\]
Pop Quiz!

• Is the number of equations equal to the number of unknowns?

• No! The number of equations is balanced by the unknown line sink strengths $\sigma$, but the head in the caisson, $\phi_c$, is also unknown.

• Need one extra equation. Let the total discharge of a well with $P$ arms be $Q_w$.

$$Q_w = \sum_{j=1}^{P} Q_j = \sum_{j=1}^{P} \sum_{m=1}^{M} \sigma_{m,j} L_{m,j}$$
Pop Quiz, continued

• Are our equations (**) linear in terms of the unknown strengths $\sigma$?
• No! The left hand sides are, but the right hand side contains $Q_m^2$, and $Q_m$ is linear in terms of the $\sigma$’s
• Solve iteratively, given an initial guess as to inflow along the arms.
Comparison to 3D: Horiz. Well

- center at (0,0); length = 60m
- elevation = 3m, radius = 0.15m
- total discharge = 12,000 m³/d
- 10 line sinks, 6m each
- $\phi_0 = 24m$ at (60,0)
- $k = 150$ m/d; unconfined
- “aquifer” = 12 layers; well in layer 8
- constant transmissivity in layer 1
- no head loss, etc.
Layer 1
Head (phreatic sfc)
Dashed - ML
Solid - 3D

Layer 8
Head at well
Dashed - ML
Solid - 3D
Streamlines
Dashed - ML
Solid - 3D

Cross section: heads
Dashed - ML
Solid - 3D
Comparison to 3D: RCW

- center at (0,0); 5 arms, length of each = 60m
- elevation = 3m, radius of each arm = 0.15m
- radius of caisson = 3m
- total discharge = 60,000 m³/d
- 10 line sinks per arm, 6m each
- $\phi_0 = 24$m at (100,0)
- $k = 150$ m/d; unconfined
- “aquifer” = 12 layers; well in layer 8
- constant transmissivity in layer 1
Layer 1
Head (phreatic sfc)
Dashed - ML
Solid - 3D

Layer 8
Head at well
Dashed - ML
Solid - 3D
Application

• (Slightly) new well geometry, aquifer properties

• Add laterals; investigate increase in well yield as a function of length of laterals and friction coefficient in Darcy-Weisbach equation
Well / Aquifer

- center at (0,0); 3 arms, length of each = 60m
- elevation = 3m, radius of each arm = 0.15m
- radius of caisson = 3m; $f = 0.02$
- specified head in caisson $\phi_c = 20m$
- $\phi_0 = 24m$ at (200,0)
- $k_h = 200$ m/d in bottom 10m; 100 m/d above
- $k_v = 60$ m/d
- “aquifer” = 18 layers; well in layer 14
- $\Rightarrow$ RESULT: $Q = 22,400$ m$^3$/d
Modifications to well

- Add 3 arms in layer 9; skewed position
- Maintain $\phi_c = 20$ m, change lengths
Summary

• Multilayer modeling of radial collector wells rules!

• Vertical stratification / anisotropy

• Boundary conditions along arm included
  – (friction factor has significant effect on well yield)

• Can be combined with regional 2D model